

SHARP BOUNDS FOR THE SECOND SEIFFERT MEAN IN TERMS OF POWER MEANS

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ABSTRACT. For $a, b > 0$ with $a \neq b$, let $T(a, b)$ denote the second Seiffert mean defined by

$$T(a, b) = \frac{a - b}{2 \arctan \frac{a-b}{a+b}}$$

and $A_r(a, b)$ denote the r -order power mean. We present the sharp bounds for the second Seiffert mean in terms of power means:

$$A_{p_1}(a, b) < T(a, b) \leq A_{p_2}(a, b),$$

where $p_1 = \log_{\pi/2} 2$ and $p_2 = 5/3$ can not be improved.

1. INTRODUCTION

Throughout the paper, we assume that $a, b > 0$ with $a \neq b$. The power mean of order r of the positive real numbers a and b is defined by

$$A_r = A_r(a, b) = \left(\frac{a^r + b^r}{2} \right)^{1/r} \text{ if } r \neq 0 \text{ and } A_0 = A_0(a, b) = \sqrt{ab}.$$

It is well-known that the function $r \mapsto A_r(a, b)$ is continuous and strictly increasing on \mathbb{R} (see [1]). As special cases, the arithmetic mean, geometric mean and quadratic mean are $A = A(a, b) = A_1(a, b)$, $G = G(a, b) = A_0(a, b)$ and $Q = Q(a, b) = A_2(a, b)$, respectively.

The Lehmer mean of order r of the positive real numbers a and b is defined as

$$L_r = L_r(a, b) = \frac{a^{r+1} + b^{r+1}}{a^r + b^r}$$

(see [11]). It is seen that the function $r \mapsto L_r(a, b)$ is continuous and strictly increasing on \mathbb{R} . In particular, $L_0 = A$, $L_1 = C$ are the arithmetic mean, contra-harmonic mean, respectively. Clearly, Lehmer mean can be expressed by power means as $L_r = A_{r+1}^{r+1} A_r^{-r}$.

The first Seiffert mean [17] is defined by

$$P = P(a, b) = \frac{a - b}{2 \arcsin \frac{a-b}{a+b}}.$$

Many remarkable inequalities for P can be found in the literature [10], [14], [7], [15], [13], [2], [9], [19], [20], [3], [12]. Here we mention that the following sharp bounds for

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the first Seiffert mean P in terms of power means proved by Jagers [10] and Hästö [8]:

$$(1.1) \quad A_{\log_{\pi} 2}(a, b) < P(a, b) < A_{2/3}(a, b).$$

In 1995, Seiffert [18] defined his second mean as

$$T = T(a, b) = \frac{a - b}{2 \arctan \frac{a-b}{a+b}}$$

and proved that

$$(1.2) \quad A < T < Q.$$

Sándor [16, pp. 265-267] showed that by a transformation of arguments, the mean T can be reduced to the mean P :

$$T(a, b) = P(x, y),$$

where

$$(1.3) \quad x = \frac{\sqrt{2(a^2 + b^2)} + a - b}{2}, \quad y = \frac{\sqrt{2(a^2 + b^2)} - a + b}{2},$$

which implies

$$A(x, y) = Q(a, b), \quad G(x, y) = A(a, b).$$

Therefore, by using the transformations (1.3), the following transformations of means will be true:

$$G \rightarrow A, \quad A \rightarrow Q, \quad P \rightarrow T.$$

Thus, from the known inequalities involving P , A , G he easily obtained corresponding ones involving T , Q , A , for example, (1.2) and the following inequalities:

$$(1.4) \quad Q^{2/3} A^{1/3} < Q^{1/3} \left(\frac{Q + A}{2} \right)^{2/3} < T < \frac{2Q + A}{3}.$$

Recently, Chu et al. in [4] proved the double inequality

$$(1.5) \quad p_1 Q + (1 - p_1) A < T < q_1 Q + (1 - q_1) A$$

holds if and only if $p_1 \leq (\sqrt{2} + 1)(4 - \pi)/\pi$, $q_1 \geq 2/3$, which shows that the constant $2/3$ of the third inequality in (1.4) is the best.

Very recently, Witkowski [21] used some geometric ideas to prove a series of inequalities involving T , Q , A , such as

$$(1.6) \quad A < T < \frac{4}{\pi} A,$$

$$(1.7) \quad \frac{2\sqrt{2}}{\pi} Q < T < Q,$$

$$(1.8) \quad (1 - r_1)Q + r_1 A < T < \frac{2Q + A}{3},$$

where $r_1 = \frac{2(\pi - 2\sqrt{2})}{(2 - \sqrt{2})\pi} = 0.340341385\dots$. It is obvious that (1.8) is actually (1.5).

In 2010, Wang et al. [20] presented the optimal upper and lower Lehmer mean bounds for T as follows:

$$(1.9) \quad L_0 < T < L_{1/3}.$$

In [5], Chu et al. demonstrated that the double inequality
 (1.10) $C(p_2 a + (1 - p_2)b, p_2 b + (1 - p_2)a) < T(a, b) < C(q_2 a + (1 - q_2)b, q_2 b + (1 - q_2)a)$
 if and only if $p_2 \leq \left(1 + \sqrt{4/\pi - 1}\right)/2$, $q_2 \geq (3 + \sqrt{3})/6$.

It is interesting and useful to evaluate the second Seiffert mean T by power means A_p . Until recently, the inequalities (1.2) has improved by Constin and Toader [6] as

$$(1.11) \quad N < A_{3/2} < T < Q,$$

where N is the Neuman-Sándor mean defined in [13] by

$$N = N(a, b) = \frac{a - b}{2 \operatorname{arcsinh} \frac{a-b}{a+b}}.$$

Up to now, this may be the best result for the bounds for the second Seiffert mean in terms of power means. For this reason, we are going to find the best $p \in (3/2, 2)$ such that the inequality

$$(1.12) \quad T(a, b) < A_p(a, b)$$

or its reverse inequality holds in this paper.

Our main results are the following

Theorem 1.1. *The inequality (1.12) if and only if $p \geq p_2 = 5/3$. Moreover, we have*

$$(1.13) \quad \alpha_1 A_{5/3}(a, b) < T(a, b) < \alpha_2 A_{5/3}(a, b),$$

where $\alpha_1 = 2^{8/5} \pi^{-1} = 0.96494\dots$ and $\alpha_2 = 1$ are the best possible constants.

Theorem 1.2. *The inequality (1.12) is reversed if and only if $p \leq p_1 = \log_{\pi/2} 2 = 1.5349\dots$. Moreover, we have*

$$(1.14) \quad \beta_1 A_{\log_{\pi/2} 2}(a, b) < T(a, b) < \beta_2 A_{\log_{\pi/2} 2}(a, b),$$

where $\beta_1 = 1$ and $\beta_2 = 1.0136\dots$ are the best possible constants.

2. LEMMAS

In order to prove our main results, we need the following lemmas.

Lemma 2.1. *Let F_p be the function defined on $(0, 1)$ by*

$$(2.1) \quad F_p(x) = \ln \frac{T(1, x)}{A_p(1, x)} = \ln \frac{1 - x}{2 \arctan \frac{1-x}{x+1}} - \frac{1}{p} \ln \left(\frac{x^p + 1}{2} \right).$$

Then we have

$$(2.2) \quad \lim_{x \rightarrow 1^-} \frac{F_p(x)}{(x-1)^2} = -\frac{1}{24}(3p-5),$$

$$(2.3) \quad F_p(0^+) = \lim_{x \rightarrow 0^+} F_p(x) = \begin{cases} \frac{1}{p} \ln 2 - \ln \frac{\pi}{2} & \text{if } p > 0, \\ \infty & \text{if } p \leq 0, \end{cases}$$

where $F_0(x) := \lim_{p \rightarrow 0} F_p(x)$.

Proof. Using power series expansion we have

$$F_p(x) = -\frac{1}{24}(3p-5)(x-1)^2 + O\left((x-1)^3\right),$$

which yields (2.2).

Direct limit calculation leads to (2.3), which proves the lemma. \square

Lemma 2.2. *Let F_p be the function defined on $(0, 1)$ by (2.1). Then F_p is strictly increasing on $(0, 1)$ if and only if $p \geq 5/3$ and decreasing on $(0, 1)$ if and only if $p \leq 1$.*

Proof. Differentiation yields

$$(2.4) \quad F'_p(x) = \frac{x^{p-1} + 1}{x(1-x)(x^p + 1) \arctan \frac{1-x}{x+1}} f_1(x),$$

where

$$(2.5) \quad f_1(x) = \frac{(1-x)(x^p + 1)}{(x^2 + 1)(x^{p-1} + 1)} - \arctan \frac{1-x}{x+1}.$$

Differentiation again leads to

$$(2.6) \quad f'_1(x) = -\frac{x(1-x)}{(x^2 + 1)^2 (x^{p-1} + 1)^2} f_2(x),$$

where

$$(2.7) \quad f_2(x) = ((1-p)x^p + (p+1)x^{p-1} - 2x^{2p-3} - (p+1)x^{p-2} + (p-1)x^{p-3} + 2).$$

(i) We now prove that F_p is strictly increasing on $(0, 1)$ if and only if $p \geq 5/3$. From (2.4) it is seen that $\operatorname{sgn} F'_p(x) = \operatorname{sgn} f_1(x)$ for $x \in (0, 1)$, so it suffices to prove that $f_1(x) > 0$ for $x \in (0, 1)$ if and only if $p \geq 5/3$.

Necessity. If $f_1(x) > 0$ for $x \in (0, 1)$ then there must be $\lim_{x \rightarrow 1^-} (1-x)^{-3} f_1(x) \geq 0$. Application of L'Hospital rule leads to

$$\lim_{x \rightarrow 1^-} \frac{f_1(x)}{(1-x)^3} = \lim_{x \rightarrow 1^-} \frac{\frac{(1-x)(x^p+1)}{(x^2+1)(x^{p-1}+1)} - \arctan \frac{1-x}{x+1}}{(1-x)^3} = \frac{1}{24}(3p-5),$$

and so we have $p \geq 5/3$.

Sufficiency. We now prove $f_1(x) > 0$ for $x \in (0, 1)$ if $p \geq 5/3$. As mentioned previous, the function

$$p \mapsto L_{p-1}(1, x) = \frac{x^p + 1}{x^{p-1} + 1}$$

is increasing on \mathbb{R} , it is enough to show that $f_1(x) > 0$ for $x \in (0, 1)$ when $p = 5/3$. In this case, we have

$$3x^{4/3}f_2(x) = -2x^3 + 8x^2 - 6x^{5/3} + 6x^{4/3} - 8x + 2.$$

Factoring yields

$$3x^{4/3}f_2(x) = 2(1 - \sqrt[3]{x})^3(x^{2/3} + 1)(x^{4/3} + 3x + 5x^{2/3} + 3x^{1/3} + 1) > 0.$$

It follows from (2.6) that $f'_1(x) < 0$, that is, the function f_1 is decreasing on $(0, 1)$. Hence for $x \in (0, 1)$ we have $f_1(x) > f_1(1) = 0$, which proves the sufficiency.

(ii) We next prove that F_p is strictly decreasing on $(0, 1)$ if and only if $p \leq 1$. Similarly, it suffices to show that $f_1(x) < 0$ for $x \in (0, 1)$ if and only if $p \leq 1$.

Necessity. If $f_1(x) < 0$ for $x \in (0, 1)$ then we have

$$\lim_{x \rightarrow 0^+} f_1(x) = \begin{cases} 1 - \frac{\pi}{4} > 0 & \text{if } p > 1, \\ \frac{1}{2} - \frac{\pi}{4} < 0 & \text{if } p = 1, \\ -\frac{\pi}{4} & \text{if } p < 1 \end{cases} \leq 0,$$

which yields $p \leq 1$.

Sufficiency. We prove $f_1(x) < 0$ for $x \in (0, 1)$ if $p \leq 1$. Due to the monotonicity of the function $p \mapsto L_{p-1}(1, x)$, it suffices to demonstrate $f_1(x) < 0$ for $x \in (0, 1)$ when $p = 1$. In this case, we have $f_2(x) = 4 - 4x^{-1} < 0$, then $f'_1(x) > 0$, and then for $x \in (0, 1)$ we have $f_1(x) < f_1(1) = 0$, which proves the sufficiency and the proof of this lemma is finished. \square

Lemma 2.3. Let f_3 be the function defined on $(0, 1)$ by

$$(2.8) \quad f_3(x) = -p(p-1)x^3 + (p-1)(p+1)x^2 - 2(2p-3)x^p - (p+1)(p-2)x + (p-1)(p-3)$$

Then f_3 is strictly increasing on $(0, 1)$ if $p \in (1, 5/3)$.

Proof. Differentiation yields

$$(2.9) \quad f'_3(x) = -3p(p-1)x^2 + 2(p-1)(p+1)x - 2p(2p-3)x^{p-1} - (p+1)(p-2).$$

Note that $1 < p < 5/3$, using basic inequality for means

$$x^{p-1} \leq (p-1)x + (2-p) \quad (x > 0)$$

to the last member of the third term in (2.9) we have

$$\begin{aligned} f'_3(x) &\geq -3p(p-1)x^2 + 2(p-1)(p+1)x \\ &\quad - 2p(2p-3)((p-1)x + (2-p)) - (p+1)(p-2) \\ &= -3p(p-1)x^2 - 2(p-1)(2p^2 - 4p - 1)x + (p-2)(4p^2 - 7p - 1) \\ &: = f_4(x). \end{aligned}$$

Thus, in order to prove $f'_3(x) > 0$, it needs to show that $f_4(x) > 0$ for $x \in (0, 1)$.

Since $f''_4(x) = -6p(p-1) < 0$ and for $p \in (1, 5/3)$

$$f_4(0^+) = (p-2) \left(p - \frac{\sqrt{65+7}}{8} \right) \left(p + \frac{\sqrt{65-7}}{8} \right) > 0,$$

$$f_4(1) = 6p \left(\frac{5}{3} - p \right) > 0,$$

application of properties of concave functions yields for $x \in (0, 1)$

$$f_4(x) > (1-x)f_4(0^+) + xf_4(1) > 0,$$

which completes the proof. \square

Lemma 2.4. Let $p \in (1, 5/3)$ and let the function $x \mapsto F_p(x)$ be defined on $(0, 1)$ by (2.1). Then the equation $f_1(x) = 0$ has a unique solution x_3 such that F_p is increasing on $(0, x_3)$ and decreasing on $(x_3, 1)$, where $f_1(x)$ is defined by (2.5).

Proof. Differentiating $f_2(x)$ defined by (2.7) gives

$$(2.10) \quad x^{4-p} f'_2(x) = f_3(x),$$

where $f_3(x)$ is defined by (2.8).

Because that f_3 is strictly increasing on $(0, 1)$ if $p \in (1, 5/3)$ by Lemma (2.3) and note that

$$f_3(0^+) = (p-1)(p-3) < 0, \quad f_3(1) = 2(5-3p) > 0,$$

there is a unique $x_1 \in (0, 1)$ such that $f_3(x) < 0$ for $x \in (0, x_1)$ and $f_3(x) > 0$ for $x \in (x_1, 1)$. Then it is seen from (2.10) that f_2 is decreasing on $(0, x_1)$ and increasing on $(x_1, 1)$, which yields $f_2(x) < f_2(1) = 0$ for $x \in (x_1, 1)$. This together with $\text{sgn } f_2(0^+) = \text{sgn}(p-1) > 0$ reveals that there exists a unique $x_2 \in (0, x_1)$ such that $f_2(x) > 0$ for $x \in (0, x_2)$ and $f_2(x) < 0$ for $x \in (x_2, 1)$. It follows from (2.6) that f_1 is decreasing on $(0, x_2)$ and increasing on $(x_2, 1)$, and therefore $f_1(x) < f_1(1) = 0$ for $x \in (x_2, 1)$, which in combination with $f_1(0^+) = 1 - \frac{1}{4}\pi > 0$ indicates that there is a unique $x_3 \in (0, x_2)$ such that $f_1(x) > 0$ for $x \in (0, x_3)$ and $f_1(x) < 0$ for $x \in (x_3, 1)$. By (2.4) it is easy to see that the function $x \mapsto F_p(x)$ is increasing on $(0, x_3)$ and decreasing on $(x_3, 1)$, which proves the lemma. \square

3. PROOFS OF MAIN RESULTS

Based on the lemmas in the above section, we can easily prove our main results.

Proof of Theorem 1.1. By symmetry, we assume that $a > b > 0$. Then inequality (1.12) is equivalent to

$$(3.1) \quad \ln T(1, x) - \ln A_p(1, x) = F_p(x) < 0,$$

where $x = b/a \in (0, 1)$. Now we prove the inequality (3.1) holds for all $x \in (0, 1)$ if and only if $p \geq 5/3$.

Necessity. If inequality (3.1) holds, then by Lemma 2.1 we have

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{F_p(x)}{(x-1)^2} = -\frac{1}{24}(3p-5) \leq 0, \\ \lim_{x \rightarrow 0^+} F_p(x) = \frac{1}{p} \ln 2 - \ln \frac{\pi}{2} \leq 0 \text{ if } p > 0, \end{cases}$$

which yields $p \geq 5/3$.

Sufficiency. Suppose that $p \geq 5/3$. It follows from Lemma 2.2 that $F_p(x) < F_p(1) = 0$ for $x \in (0, 1)$, which proves the sufficiency.

Using the monotonicity of the function $x \mapsto F_{5/3}(x)$ on $(0, 1)$, we have

$$\ln(2^{8/5}\pi^{-1}) = F_{5/3}(0^+) < F_{5/3}(x) < F_{5/3}(1^-) = 0,$$

which implies (1.13).

Thus the proof of Theorem 1.1 is finished. \square

Proof of Theorem 1.2. Clearly, the reverse inequality of (1.12) is equivalent to

$$(3.2) \quad \ln T(1, x) - \ln A_p(1, x) = F_p(x) > 0,$$

where $x = b/a \in (0, 1)$. Now we show that the inequality (3.2) holds for all $x \in (0, 1)$ if and only if $p \leq \log_{\pi/2} 2$.

Necessity. The condition $p \leq \log_{\pi/2} 2$ is necessary. Indeed, if inequality (3.2) holds, then we have

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{F_p(x)}{(x-1)^2} = -\frac{1}{24}(3p-5) \geq 0, \\ \lim_{x \rightarrow 0^+} F_p(x) = \frac{1}{p} \ln 2 - \ln \frac{\pi}{2} \geq 0 \text{ if } p > 0 \end{cases}$$

or

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{F_p(x)}{(x-1)^2} = -\frac{1}{24}(3p-5) \geq 0, \\ \lim_{x \rightarrow 0^+} F_p(x) = \infty \text{ if } p \leq 0. \end{cases}$$

Solving the above inequalities leads to $p \leq \log_{\pi/2} 2$.

Sufficiency. The condition $p \leq \log_{\pi/2} 2$ is also sufficient. Since the function $r \mapsto A_r(1, x)$ is increasing, so the function $p \mapsto F_p(x)$ is decreasing, thus it suffices to show that $F_p(x) > 0$ for all $x \in (0, 1)$ if $p = p_1 = \log_{\pi/2} 2$.

Lemma 2.4 reveals that for $p \in (1, 5/3)$ there is a unique x_3 to satisfy

$$(3.3) \quad f_1(x_3) = \frac{(1-x_3)(x_3^p+1)}{(x_3^2+1)(x_3^{p-1}+1)} - \arctan \frac{1-x_3}{x_3+1} = 0$$

such that the function $x \mapsto F_p(x)$ is strictly increasing on $(0, x_3)$ and strictly decreasing on $(x_3, 1)$. It is acquired that for $p_1 = \log_{\pi/2} 2 \in (1, 5/3)$

$$\begin{aligned} 0 &= F_{p_1}(0^+) < F_{p_1}(x) \leq F_{p_1}(x_3) \\ 0 &= F_{p_1}(1) < F_{p_1}(x_3) \leq F_{p_1}(x_3), \end{aligned}$$

which leads to

$$A_{p_1}(1, x) < T(1, x) < (\exp F_p(x_3)) A_{p_1}(1, x).$$

Solving the equation (3.3) for x_3 by mathematical computation software we find that $x_3 \in (0.186930110570624, 0.186930110570625)$, and then

$$\beta_2 = \exp(F_{p_1}(x_3)) \approx 1.0136,$$

which proves the sufficiency and inequalities of (1.14). \square

4. REMARKS

Remark 4.1. From the proof of Lemma 2.2, it is seen that $f_1(x) > 0$ if and only if $p \geq 5/3$, which implies that the inequality

$$T(1, x) = \frac{x-1}{2 \arctan \frac{x-1}{x+1}} > \frac{(x^2+1)(x^{p-1}+1)}{2(x^p+1)}$$

holds if and only $p \geq 5/3$. In a similar way, the inequality

$$T(1, x) < \frac{(x^2+1)(x^{p-1}+1)}{2(x^p+1)}$$

is valid if and only if $p \leq 1$. The results can be restated as a corollary.

Corollary 1. The inequalities

$$(4.1) \quad \frac{(a^2+b^2)(a^{2/3}+b^{2/3})}{2(a^{5/3}+b^{5/3})} < T(a, b) < \frac{a^2+b^2}{a+b}$$

with the best constants $5/3$ and 1 , and the function

$$p \mapsto \frac{(a^2+b^2)(a^{p-1}+b^{p-1})}{2(a^p+b^p)}$$

is decreasing.

In particular, putting $p = 1, 1/2, \dots, \rightarrow -\infty$ and $5/3, 2, \dots, \rightarrow \infty$ we get

$$\begin{aligned} \frac{a^2 + b^2}{2 \max(a, b)} &< \dots < \frac{a + b}{2} < \frac{(a^2 + b^2)(a^{2/3} + b^{2/3})}{2(a^{5/3} + b^{5/3})} < T(a, b) \\ &< \frac{a^2 + b^2}{a + b} < \frac{a^2 + b^2}{2\sqrt{ab}} < \dots < \frac{a^2 + b^2}{2 \min(a, b)}. \end{aligned}$$

Remark 4.2. Using the monotonicity of the function defined on $(0, 1)$ by

$$F_p(x) = \ln \frac{T(1, x)}{A_p(1, x)}$$

given in Lemma 2.2, we can obtain a Fan Ky type inequality but omit the further details of the proof.

Corollary 2. Let $a_1, a_2, b_1, b_2 > 0$ with $a_1/b_1 < a_2/b_2 < 1$. Then the following Fan Ky type inequality

$$\frac{T(a_1, b_1)}{T(a_2, b_2)} < \frac{A_p(a_1, b_1)}{A_p(a_2, b_2)}$$

holds if $p \geq 5/3$. It is reversed if $p \leq 1$.

Remark 4.3. As sharp upper bounds for the second Seiffert mean, we have the following relations:

$$(4.2) \quad T < \frac{2Q + A}{3} < A_{5/3} < L_{1/3}.$$

In fact, it has been shown in [22, Conclusion 1] that the function $r \mapsto A_r$ is strictly log-concave on $[0, \infty)$, and therefore

$$A_{5/3}^{3/4} A_{1/3}^{1/4} < A_{\frac{3}{4} \cdot \frac{5}{3} + \frac{1}{4} \cdot \frac{1}{3}} = A_{4/3},$$

which is equivalent with the third inequality in (4.2). Now we prove the second one. Assume that $a > b > 0$ and set $(a/b)^{1/3} = x \in (0, 1)$. Then the inequality in question is equivalent to

$$D(x) := \ln \frac{2\sqrt{\frac{x^6+1}{2}} + \frac{x^3+1}{2}}{3} - \ln \left(\frac{x^5+1}{2} \right)^{3/5} < 0.$$

Differentiating $D(x)$ yields

$$D'(x) = \frac{3x^2(1-x)}{(x^5+1)\left(x^3\sqrt{\frac{1}{2}x^6+\frac{1}{2}} + \sqrt{\frac{1}{2}x^6+\frac{1}{2}} + 2x^6+2\right)} D_1(x),$$

where

$$\begin{aligned} D_1(x) &= (1+x)\sqrt{\frac{1}{2}x^6+\frac{1}{2}} - 2x^2 = \frac{\left((1+x)\sqrt{\frac{1}{2}x^6+\frac{1}{2}}\right)^2 - (2x^2)^2}{(1+x)\sqrt{\frac{1}{2}x^6+\frac{1}{2}} + 2x^2} \\ &= \frac{(x-1)^2(x^6+4x^5+8x^4+12x^3+8x^2+4x+1)}{2(1+x)\sqrt{\frac{1}{2}x^6+\frac{1}{2}} + 2x^2} > 0. \end{aligned}$$

Hence, $D'(x) > 0$ for $x \in (0, 1)$, then $D(x) < D(1) = 0$.

Remark 4.4. By Theorem 1.1 and 1.2, the inequalities (1.2) and (1.11) can be improved as

$$(4.3) \quad N < A_{3/2} < A_{\log_{\pi/2} 2} < T < A_{5/3} < A_2.$$

In our forthcoming paper, we shall establish the sharp bounds for the Neuman-Sándor mean in terms of power means as follows:

$$(4.4) \quad A_{p_0} < N < A_{4/3},$$

where $p_0 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})} = 1.2228\dots$

Thus the chain of inequalities for bivariate means given in [6, (1)] can be refined as a more nice one:

$$(4.5) \quad A_0 < L < A_{1/3} < P < A_{2/3} < I < A_{3/3} < N < A_{4/3} < T < A_{5/3},$$

where L, P, I, N, T are the logarithmic mean, the first Seiffert mean, identric mean, Neuman-Sándor mean, the second Seiffert mean, respectively.

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